

3.2 Big-Step Structural Operational Semantics (Big-Step SOS)

Known also under the names *natural semantics*, *relational semantics*, and *evaluation semantics*, big-step structural operational semantics, or *big-step SOS* for short, is the most “denotational” of the operational semantics: one can view big-step definitions as definitions of functions, or more generally of relations, interpreting each language construct in an appropriate domain. Big-step semantics is so easy and natural to use, that one is strongly encouraged to use it whenever possible. Unfortunately, as discussed in Section 3.10, big-step semantics has a series of limitations making it inconvenient or impossible to use in many situations, such as when defining control-intensive language features, or non-deterministic ones, or concurrency.

A big-step SOS of a programming language or calculus is given as a formal *proof system* (see Section 2.1.5). The *big-step SOS sequents* are relations over configurations, typically written $C \Rightarrow R$ or $C \Downarrow R$, with the meaning that R is the configuration obtained after the (complete) evaluation of C . In this book we prefer the notation $C \Downarrow R$. A *big-step SOS rule* therefore has the form

$$\frac{C_1 \Downarrow R_1 \quad C_2 \Downarrow R_2 \quad \dots \quad C_n \Downarrow R_n}{C_0 \Downarrow R_0} \quad [\text{if condition}]$$

where $C_0, C_1, C_2, \dots, C_n$ are configurations holding fragments of program together with all the needed semantic components, where $R_0, R_1, R_2, \dots, R_n$ are *result configurations*, or *irreducible configurations*, i.e., configurations which cannot be reduced anymore, and where *condition* is an optional *side condition*; as discussed in Section 2.1.5, the role of side conditions is to filter out undesirable instances of the rule.

A big-step semantics compositionally describes how final evaluation results of language constructs can be obtained by combining the evaluation results of their syntactic counterparts (subexpressions, etc.). For example, the big-step semantics of IMP’s addition is

$$\frac{\langle a_1, \sigma \rangle \Downarrow \langle i_1 \rangle \quad \langle a_2, \sigma \rangle \Downarrow \langle i_2 \rangle}{\langle a_1 + a_2, \sigma \rangle \Downarrow \langle i_1 +_{int} i_2 \rangle}$$

Here, the meaning of a relation $\langle a, \sigma \rangle \Downarrow \langle i \rangle$ is that arithmetic expression a is evaluated in state σ to integer i . If expression evaluation has side-effects, then one has to also include a state in the right configurations, so they become of the form $\langle i, \sigma \rangle$ instead of $\langle i \rangle$, as discussed in Section 3.10.

It is common in big-step semantics to not wrap single values in configurations, that is, to write $\langle a, \sigma \rangle \Downarrow i$ instead of $\langle a, \sigma \rangle \Downarrow \langle i \rangle$ and similarly for all the other sequents. Also, while the angle-bracket-and-comma notation $\langle \text{code}, \text{state}, \dots \rangle$ is common for configurations, it is not enforced; some prefer to use a square or curly bracket notation of the form $[\text{code}, \text{state}, \dots]$ or $\{\text{code}, \text{state}, \dots\}$, or the simple tuple notation $(\text{code}, \text{state}, \dots)$, or even to use a different (from comma) symbol to separate the various configuration ingredients, e.g., $\langle \text{code} \mid \text{state} \mid \dots \rangle$, etc. Moreover, we may even encounter in the literature sequent notations of the form $\sigma \vdash a \Rightarrow i$ instead of $\langle a, \sigma \rangle \Downarrow \langle i \rangle$, as well as variants of sequent notations that prefer to move various semantic components from the configurations into special, sometimes rather informal, decorations of the symbols \Downarrow , \vdash and/or \Rightarrow .

For the sake of a uniform notation, in particular when transitioning from languages whose expressions have no side effects to languages whose expressions do have side effects (as we do in Section 3.10), we prefer to always write big-step sequents as $C \Downarrow R$, and always use the angle brackets to surround both configurations involved. This solution is the most general; for example, any additional semantic data or labels that one may need in a big-step definition can be uniformly included as additional components in the configurations (the left ones, or the right ones, or both).

sorts:
Configuration

operations:
 $\langle -, - \rangle : AExp \times State \rightarrow Configuration$
 $\langle - \rangle : Int \rightarrow Configuration$
 $\langle -, - \rangle : BExp \times State \rightarrow Configuration$
 $\langle - \rangle : Bool \rightarrow Configuration$
 $\langle -, - \rangle : Stmt \times State \rightarrow Configuration$
 $\langle - \rangle : State \rightarrow Configuration$
 $\langle - \rangle : Pgm \rightarrow Configuration$

Figure 3.6: IMP big-step configurations as an algebraic signature.

3.2.1 IMP Configurations for Big-Step SOS

For the big-step semantics of the simple language IMP, we only need very simple configurations. We follow the comma-and-angle-bracket notational convention, that is, we separate the configuration components by commas and then enclose the entire list with angle brackets. For example, $\langle a, \sigma \rangle$ is a configuration containing an arithmetic expression a and a state σ , and $\langle b, \sigma \rangle$ is a configuration containing a Boolean expression b and a state σ . Some configurations may not need a state while others may not need the code. For example, $\langle i \rangle$ is a configuration holding only the integer number i that can be obtained as a result of evaluating an arithmetic expression, while $\langle \sigma \rangle$ is a configuration holding only one state σ that can be obtained after evaluating a statement. Configurations can therefore be of different types and need not necessarily have the same number of components. Here are all the configuration types needed for the big-step semantics of IMP:

- $\langle a, \sigma \rangle$ grouping arithmetic expressions a and states σ ;
- $\langle i \rangle$ holding integers i ;
- $\langle b, \sigma \rangle$ grouping Boolean expressions b and states σ ;
- $\langle t \rangle$ holding truth values $t \in \{true, false\}$;
- $\langle s, \sigma \rangle$ grouping statements s and states σ ;
- $\langle \sigma \rangle$ holding states σ ;
- $\langle p \rangle$ holding programs p .

IMP Big-Step SOS Configurations as an Algebraic Signature

The configurations above were defined rather informally as tuples of syntax and/or states. There are many ways to rigorously formalize them, all building upon some formal definition of state (besides IMP syntax). Since we have already defined states as partial finite-domain functions (Section 3.1.2) and have already shown how partial finite-domain functions can be formalized as algebraic specifications (Section 2.4.6), we also formalize configurations algebraically.

Figure 3.6 shows an algebraic signature defining the IMP configurations needed for the subsequent big-step operational semantics. For simplicity, we preferred to explicitly define each type of needed configuration.

Consequently, our configurations definition in Figure 3.6 may be more verbose than an alternative polymorphic definition, but we believe that it is clearer for this simple language. We assumed that the sorts $AExp$, $BExp$, $Stmt$, Pgm and $State$ come from algebraic definitions of the IMP syntax and state, like those in Sections 3.1.1 and 3.1.2; recall that the latter adapted the algebraic definition of partial functions in Section 2.4.6 (see Figure 2.7) as explained in Section 3.1.2.

3.2.2 The Big-Step SOS Rules of IMP

Figure 3.7 shows all the rules in our IMP big-step operational semantics proof system. Recall that the role of a proof system is to derive sequents, or facts. The facts that our proof system will derive have the forms $\langle a, \sigma \rangle \Downarrow \langle i \rangle$, $\langle b, \sigma \rangle \Downarrow \langle t \rangle$, $\langle s, \sigma \rangle \Downarrow \langle \sigma' \rangle$, and $\langle p \rangle \Downarrow \langle \sigma \rangle$ where a ranges over $AExp$, b over $BExp$, s over $Stmt$, p over Pgm , i over Int , t over $Bool$, and σ and σ' over $State$.

Informally², the meaning of derived triples of the form $\langle a, \sigma \rangle \Downarrow \langle i \rangle$ is that the arithmetic expression a evaluates/executes/transitions to the integer i in state σ ; the meaning of $\langle b, \sigma \rangle \Downarrow \langle t \rangle$ is similar but with Boolean values instead of integers. The reason for which it suffices to derive such simple facts is because the evaluation of expressions in our simple IMP language is side-effect-free. When we add the increment operation $++x$ in Section 3.10, we will have to change the big-step semantics to work with 4-tuples of the form $\langle a, \sigma \rangle \Downarrow \langle i, \sigma' \rangle$ and $\langle b, \sigma \rangle \Downarrow \langle t, \sigma' \rangle$ instead. The meaning of $\langle s, \sigma \rangle \Downarrow \langle \sigma' \rangle$ is that the statement s takes state σ to state σ' . Finally, the meaning of pairs $\langle p \rangle \Downarrow \langle \sigma \rangle$ is that the program p yields state σ when executed in the initial state.

In the case of our simple IMP language, the transition relation is going to be *deterministic*, in the sense that $i_1 = i_2$ whenever $\langle a, \sigma \rangle \Downarrow \langle i_1 \rangle$ and $\langle a, \sigma \rangle \Downarrow \langle i_2 \rangle$ can be deduced (and similarly for Boolean expressions, statements, and programs). However, in the context of non-deterministic languages, triples $\langle a, \sigma \rangle \Downarrow \langle i \rangle$ state that a *may* evaluate to i in state σ , but it may also evaluate to other integers (and similarly for Boolean expressions, statements, and programs).

The proof system in Figure 3.7 contains one or two rules for each language construct, capturing its intended evaluation relation. Recall from Section 2.1.5 that proof rules are in fact *rule schemas*, that is, they correspond to (recursively enumerable) sets of *rule instances*, one for each concrete instance of the rule *parameters* (i.e., a, b, σ , etc.). We next discuss each of the rules in Figure 3.7.

The rules (BIGSTEP-INT) and (BIGSTEP-LOOKUP) define the obvious semantics of integers and of variable lookup; these rules have no premises because integers and variables are atomic expressions, so one does not need to evaluate any other subexpression in order to evaluate them. The rule (BIGSTEP-ADD) has already been discussed at the beginning of Section 3.2, and (BIGSTEP-DIV) is similar. Note that the rules (BIGSTEP-LOOKUP) and (BIGSTEP-DIV) have side conditions. We chose not to short-circuit the division operation when a_1 evaluates to 0. Consequently, no matter whether a_1 evaluates to 0 or not, a_2 is still expected to produce a correct value in order for the rule (BIGSTEP-DIV) to be applicable (e.g., a_2 cannot perform a division by 0).

Before we continue with the remaining rules, let us clarify, using concrete examples, what it means for rule schemas to admit multiple instances and how these can be used to derive proofs. For example, a possible instance of rule (BIGSTEP-DIV) can be the following (assume that $x, y \in Id$):

$$\frac{\langle x, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 8 \rangle \quad \langle 2, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 2 \rangle}{\langle x / 2, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 4 \rangle}$$

²Formal definitions of these concepts can only be given after one has a formal language definition. We formally define the notions of evaluation and termination in the context of the IMP language in Definition 20.

$\langle i, \sigma \rangle \Downarrow \langle i \rangle$	(BIGSTEP-INT)
$\langle x, \sigma \rangle \Downarrow \langle \sigma(x) \rangle \quad \text{if } \sigma(x) \neq \perp$	(BIGSTEP-LOOKUP)
$\frac{\langle a_1, \sigma \rangle \Downarrow \langle i_1 \rangle \quad \langle a_2, \sigma \rangle \Downarrow \langle i_2 \rangle}{\langle a_1 + a_2, \sigma \rangle \Downarrow \langle i_1 +_{int} i_2 \rangle}$	(BIGSTEP-ADD)
$\frac{\langle a_1, \sigma \rangle \Downarrow \langle i_1 \rangle \quad \langle a_2, \sigma \rangle \Downarrow \langle i_2 \rangle}{\langle a_1 / a_2, \sigma \rangle \Downarrow \langle i_1 /_{int} i_2 \rangle} \quad \text{if } i_2 \neq 0$	(BIGSTEP-DIV)
$\langle t, \sigma \rangle \Downarrow \langle t \rangle$	(BIGSTEP-BOOL)
$\frac{\langle a_1, \sigma \rangle \Downarrow \langle i_1 \rangle \quad \langle a_2, \sigma \rangle \Downarrow \langle i_2 \rangle}{\langle a_1 \leq a_2, \sigma \rangle \Downarrow \langle i_1 \leq_{int} i_2 \rangle}$	(BIGSTEP-LEQ)
$\frac{\langle b, \sigma \rangle \Downarrow \langle \text{true} \rangle}{\langle ! b, \sigma \rangle \Downarrow \langle \text{false} \rangle}$	(BIGSTEP-NOT-TRUE)
$\frac{\langle b, \sigma \rangle \Downarrow \langle \text{false} \rangle}{\langle ! b, \sigma \rangle \Downarrow \langle \text{true} \rangle}$	(BIGSTEP-NOT-FALSE)
$\frac{\langle b_1, \sigma \rangle \Downarrow \langle \text{false} \rangle}{\langle b_1 \ \&\& \ b_2, \sigma \rangle \Downarrow \langle \text{false} \rangle}$	(BIGSTEP-AND-FALSE)
$\frac{\langle b_1, \sigma \rangle \Downarrow \langle \text{true} \rangle \quad \langle b_2, \sigma \rangle \Downarrow \langle t \rangle}{\langle b_1 \ \&\& \ b_2, \sigma \rangle \Downarrow \langle t \rangle}$	(BIGSTEP-AND-TRUE)
$\langle \{\}, \sigma \rangle \Downarrow \langle \sigma \rangle$	(BIGSTEP-EMPTY-BLOCK)
$\frac{\langle s, \sigma \rangle \Downarrow \langle \sigma' \rangle}{\langle \{ s \}, \sigma \rangle \Downarrow \langle \sigma' \rangle}$	(BIGSTEP-BLOCK)
$\frac{\langle a, \sigma \rangle \Downarrow \langle i \rangle}{\langle x = a; , \sigma \rangle \Downarrow \langle \sigma[i/x] \rangle} \quad \text{if } \sigma(x) \neq \perp$	(BIGSTEP-ASGN)
$\frac{\langle s_1, \sigma \rangle \Downarrow \langle \sigma_1 \rangle \quad \langle s_2, \sigma_1 \rangle \Downarrow \langle \sigma_2 \rangle}{\langle s_1 \ s_2, \sigma \rangle \Downarrow \langle \sigma_2 \rangle}$	(BIGSTEP-SEQ)
$\frac{\langle b, \sigma \rangle \Downarrow \langle \text{true} \rangle \quad \langle s_1, \sigma \rangle \Downarrow \langle \sigma_1 \rangle}{\langle \text{if } (b) \ s_1 \ \text{else } s_2, \sigma \rangle \Downarrow \langle \sigma_1 \rangle}$	(BIGSTEP-IF-TRUE)
$\frac{\langle b, \sigma \rangle \Downarrow \langle \text{false} \rangle \quad \langle s_2, \sigma \rangle \Downarrow \langle \sigma_2 \rangle}{\langle \text{if } (b) \ s_1 \ \text{else } s_2, \sigma \rangle \Downarrow \langle \sigma_2 \rangle}$	(BIGSTEP-IF-FALSE)
$\frac{\langle b, \sigma \rangle \Downarrow \langle \text{false} \rangle}{\langle \text{while } (b) \ s, \sigma \rangle \Downarrow \langle \sigma \rangle}$	(BIGSTEP-WHILE-FALSE)
$\frac{\langle b, \sigma \rangle \Downarrow \langle \text{true} \rangle \quad \langle s \ \text{while } (b) \ s, \sigma \rangle \Downarrow \langle \sigma' \rangle}{\langle \text{while } (b) \ s, \sigma \rangle \Downarrow \langle \sigma' \rangle}$	(BIGSTEP-WHILE-TRUE)
$\frac{\langle s, xl \mapsto 0 \rangle \Downarrow \langle \sigma \rangle}{\langle \text{int } xl; s \rangle \Downarrow \langle \sigma \rangle}$	(BIGSTEP-PGM)

Figure 3.7: BIGSTEP(IMP) — Big-step SOS of IMP ($i, i_1, i_2 \in Int$; $x \in Id$; $xl \in \mathbf{List}\{Id\}$; $a, a_1, a_2 \in AExp$; $t \in Bool$; $b, b_1, b_2 \in BExp$; $s, s_1, s_2 \in Stmt$; $\sigma, \sigma', \sigma_1, \sigma_2 \in State$).

Another instance of rule (BIGSTEP-DIV) is the following, which, of course, seems problematic:

$$\frac{\langle x, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 8 \rangle \quad \langle 2, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 4 \rangle}{\langle x / 2, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 2 \rangle}$$

The rule above is indeed a correct instance of (BIGSTEP-DIV), but, however, one will never be able to infer $\langle 2, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 4 \rangle$, so this rule can never be applied in a correct inference.

Note, however, that the following is *not* an instance of (BIGSTEP-DIV), no matter what $?$ is chosen to be (\perp , or $8/\text{int } 0$, etc.):

$$\frac{\langle x, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 8 \rangle \quad \langle y, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle 0 \rangle}{\langle x / y, (x \mapsto 8, y \mapsto 0) \rangle \Downarrow \langle ? \rangle}$$

Indeed, the above does not satisfy the side condition of (BIGSTEP-DIV).

The following is a valid proof derivation, where $x, y \in Id$ and $\sigma \in State$ with $\sigma(x) = 8$ and $\sigma(y) = 0$:

$$\frac{\frac{\frac{\cdot}{\langle y, \sigma \rangle \Downarrow \langle 0 \rangle} \quad \frac{\cdot}{\langle x, \sigma \rangle \Downarrow \langle 8 \rangle}}{\langle y / x, \sigma \rangle \Downarrow \langle 0 \rangle} \quad \frac{\cdot}{\langle 2, \sigma \rangle \Downarrow \langle 2 \rangle}}{\frac{\cdot}{\langle x, \sigma \rangle \Downarrow \langle 8 \rangle} \quad \frac{\cdot}{\langle y / x + 2, \sigma \rangle \Downarrow \langle 2 \rangle}}{\langle x / (y / x + 2), \sigma \rangle \Downarrow \langle 4 \rangle}$$

The proof above can be regarded as a tree, with dots as leaves and instances of rule schemas as nodes. We call such complete (in the sense that their leaves are all dots and their nodes are correct rule instances) trees *proof trees*. This way, we have a way to mathematically *derive facts*, or *sequents*, about programs directly within their semantics. We may call the root of a proof tree the *fact (or sequent) that was proved or derived*, and the tree *its proof or derivation*.

Recall that our original intention was, for demonstration purposes, to attach various evaluation strategies to the arithmetic operations. We wanted $+$ and $/$ to be non-deterministic and \leq to be left-right sequential; a non-deterministic evaluation strategy means that the subexpressions are evaluated in any order, possibly interleaving their evaluation steps, which is different from non-deterministically picking an order and then evaluating the subexpressions sequentially in that order. As an analogy, the former corresponds to evaluating the subexpressions concurrently on a multi-threaded machine, while the latter to non-deterministically queuing the subexpressions and then evaluating them one by one on a sequential machine. The former has obviously potentially many more possible behaviors than the latter. Note that many programming languages opt for non-deterministic evaluation strategies for their expression constructs precisely to allow compilers to evaluate them in any order or even concurrently; some language manuals explicitly warn the reader not to rely on any evaluation strategy of arithmetic constructs when writing programs.

Unfortunately, big-step semantics is not appropriate for defining non-deterministic evaluation strategies, because such strategies are, by their nature, small-step. One way to attempt to do it is to work with *sets* of result configurations instead of just with result configurations and thus associate to each fragment of program in a state the set of all the results that it can non-deterministically yield. However, such an approach would significantly complicate the big-step definition, so we prefer to not do it. Moreover, since IMP has no side effects yet (we will add it side effects in Section 3.5), the non-deterministic evaluation strategies would not lead to non-deterministic results anyway.

We next discuss the big-step rules for Boolean expressions. The rule (BIGSTEP-BOOL) is similar to rule (BIGSTEP-INT), but it has only two instances, one for $t = \text{true}$ and one for $t = \text{false}$. The rule (BIGSTEP-LEQ) allows to derive Boolean sequents from arithmetic ones; although we want \leq to evaluate

its arguments from left to right, there is no need to do anything more at this stage, because expressions have no side effects in IMP; in Section 3.5 we will see how to achieve the desired evaluation strategy in the presence of side effects in expressions. The rules (BIGSTEP-NOT-TRUE) and (BIGSTEP-NOT-FALSE) are clear; they could have been combined into only one rule if we had assumed our builtin *Bool* equipped with a negation operation. Unlike the division, the conjunction has a short-circuited semantics: if the first conjunct evaluates to **false** then the entire conjunction evaluates to **false** (rule (BIGSTEP-AND-FALSE)), and if the first conjunct evaluates to **true** then the conjunction evaluates to whatever truth value the second conjunct evaluates (rule (BIGSTEP-AND-TRUE)).

The role of statements in a language is to change the program state. Consequently, the rules for statements derive triples of the form $\langle s, \sigma \rangle \Downarrow \langle \sigma' \rangle$ with the meaning that if statement s is executed in state σ and *terminates*, then the resulting state is σ' . We will shortly discuss the aspect of termination in more detail. Rule (BIGSTEP-EMPTY-BLOCK) states that $\{\}$ does nothing with the state. (BIGSTEP-BLOCK) states that the block has the same semantics as the enclosed statement, which is correct at this stage because IMP has no local variable declarations; this will change in Section 3.5. (BIGSTEP-ASGN) shows how the state σ gets updated by an assignment statement “ $x = a;$ ” after a is evaluated in state σ using the rules for arithmetic expressions discussed above. (BIGSTEP-SEQ) shows how the state updates are propagated by the sequential composition of statements, and rules (BIGSTEP-IF-TRUE) and (BIGSTEP-IF-FALSE) show how the conditional first evaluates its condition and then, depending upon the truth value of that, it either evaluates its then-branch or its else-branch, but never both. The rules giving the big-step semantics of the while loop say that if the condition evaluates to **false** then the while loop dissolves and the state stays unchanged, and if the condition evaluates to **true** then the body followed by the very same while loop is evaluated (rule (BIGSTEP-WHILE-TRUE)). Finally, (BIGSTEP-PGM) gives the semantics of programs as the semantics of their statement in a state instantiating all the declared variables to 0.

On Proof Derivations, Evaluation, and Termination

So far we have used the words “evaluation” and “termination” informally. In fact, without a formal definition of a programming language, there is no other way, but informal, to define these notions. Once one has a formal definition of a language, one can not only formally define important concepts like evaluation and termination, but can also rigorously reason about programs.

Definition 20. *Given appropriate IMP configurations C and R , the IMP big-step sequent $C \Downarrow R$ is **derivable**, written $\text{BIGSTEP}(\text{IMP}) \vdash C \Downarrow R$, iff there is some proof tree rooted in $C \Downarrow R$ which is derivable using the proof system $\text{BIGSTEP}(\text{IMP})$ in Figure 3.7. Arithmetic (resp. Boolean) expression $a \in \text{AExp}$ (resp. $b \in \text{BExp}$) **evaluates** to integer $i \in \text{Int}$ (resp. to truth value $t \in \{\text{true}, \text{false}\}$) in state $\sigma \in \text{State}$ iff $\text{BIGSTEP}(\text{IMP}) \vdash \langle a, \sigma \rangle \Downarrow \langle i \rangle$ (resp. $\text{BIGSTEP}(\text{IMP}) \vdash \langle b, \sigma \rangle \Downarrow \langle t \rangle$). Statement s **terminates** in state σ iff $\text{BIGSTEP}(\text{IMP}) \vdash \langle s, \sigma \rangle \Downarrow \langle \sigma' \rangle$ for some $\sigma' \in \text{State}$; if that is the case, then we say that s **evaluates** in state σ to state σ' , or that it **takes** state σ to state σ' . Finally, program p **terminates** iff $\text{BIGSTEP}(\text{IMP}) \vdash \langle p \rangle \Downarrow \langle \sigma \rangle$ for some $\sigma \in \text{State}$.*

There are two reasons for which an IMP statement s may not terminate in a state σ : because it may contain a loop that does not terminate, or because it performs a division by zero and thus the rule (BIGSTEP-DIV) cannot apply. In the former case, the process of proof search does not terminate, while in the second case the process of proof search terminates in principle, but with a failure to find a proof. Unfortunately, big-step semantics cannot make any distinction between the two reasons for which a proof derivation cannot be found. Hence, the termination notion in Definition 20 rather means *termination with no error*. To catch

division-by-zero within the semantics, we need to add a special *error* value that a division by zero would evaluate to, and then to propagate it through all the language constructs (see Exercise 56).

A formal definition of a language allows to also formally define what it means for the language to be deterministic and to also prove it. For example, we can prove that if an IMP program p terminates then there is a unique state σ such that $\text{BIGSTEP}(\text{IMP}) \vdash \langle p \rangle \Downarrow \langle \sigma \rangle$ (see Exercise 57).

Since each rule schema comprises a *recursively enumerable* collection of concrete instances and since we have a finite set of rule schemata, we can enumerate all the instances of all these rules. Furthermore, since proof trees built with nodes in a recursively enumerable set are themselves recursively enumerable, it follows that the set of proof trees derivable with the proof system in Figure 3.7 is recursively enumerable. In other words, we can find an algorithm that enumerates all the proof trees, in particular one that enumerates all the derivable sequents $C \Downarrow R$. By enumerating all proof trees, given a terminating IMP program p , one can eventually find the unique state σ such that $\langle p \rangle \Downarrow \langle \sigma \rangle$ is derivable. This simple-minded algorithm may take a very long time and a huge amount of resources, but it is insightful to understand that it can be done.

It can be shown that there is no algorithm, based on proof derivation like above or on anything else, which takes as input an IMP program and says whether it terminates or not (see Exercise 58). This follows from the fact that our simple language, due to its while loops and arbitrarily large integers, is Turing-complete. Thus, if one were able to decide termination of programs in our language then one would also be able to decide termination of Turing machines, contradicting one of the basic undecidable problems, the *halting problem* (see Section 2.2.1 for more on Turing machines).

An interesting observation here is that non-termination of a program corresponds to *lack of proof*, and that the latter is not decidable in many interesting logics. Indeed, in *complete* logics, that is, logics that admit a complete proof system, one can enumerate all the truths. However, in general there is not much one can do about non-truths, because the enumeration algorithm will loop forever when run on a non-truth. In decidable logics one can enumerate both truths and non-truths; clearly, decidable logics are not powerful enough for our task of defining programming languages, precisely because of the halting problem argument above.

3.2.3 Big-Step SOS in Rewrite Logic

Due to its straightforward recursive nature, big-step semantics is typically easy to represent in other formalisms and also easy to translate into interpreters for the defined languages in any programming language. (The difficulty with big-step semantics is to actually give big-step semantics to complex constructs, as illustrated and discussed in Section 3.5.) It should therefore come at no surprise to the reader that one can associate a conditional rewrite rule to each big-step rule and hereby obtain a rewrite logic theory that faithfully captures the big-step definition.

In this section we first show that any big-step operational semantics BIGSTEP can be mechanically translated into a rewrite logic theory $\mathcal{R}_{\text{BIGSTEP}}$ in such a way that the corresponding derivation relations are step-for-step equivalent, that is, $\text{BIGSTEP} \vdash C \Downarrow R$ if and only if $\mathcal{R}_{\text{BIGSTEP}} \vdash \mathcal{R}_{C \Downarrow R}$, where $\mathcal{R}_{C \Downarrow R}$ is the corresponding syntactic translation of the big-step sequent $C \Downarrow R$ into a (one-step) rewrite rule. Second, we apply our generic translation technique on the big-step operational semantics $\text{BIGSTEP}(\text{IMP})$ and obtain a rewrite logic semantics of IMP that is step-for-step equivalent to the original big-step semantics of IMP. Finally, we show how $\mathcal{R}_{\text{BIGSTEP}(\text{IMP})}$ can be seamlessly defined in Maude, thus yielding an interpreter for IMP essentially for free.

Faithful Embedding of Big-Step SOS into Rewrite Logic

To define our translation generically, we need to make some assumptions about the existence of an algebraic axiomatization of configurations. More precisely, as also explained in Section 2.1.3, we assume that for any parametric term t (which can be a configuration, a condition, etc.), the term \bar{t} is an equivalent algebraic variant of t of appropriate sort. For example, a parametric configuration C is a configuration that may possibly make use of parameters, such as $a \in AExp$, $\sigma \in State$, etc.; by equivalent algebraic variant we mean a term \bar{C} of sort *Configuration* over an appropriate signature of configurations like the one that we defined for IMP in Section 3.2.1 (see Figure 3.6); moreover, each parameter in C gets replaced by a *variable* of corresponding sort in \bar{C} . Similarly, the algebraic variant of a rule side condition is an appropriate term of sort *Bool*.

To have a formal mechanism for performing reasoning within the employed mathematical domains, which is tacitly assumed in the big step semantics (e.g., $3 +_{int} 5 = 8$, etc.), in particular for formally evaluating side conditions, we assume that the algebraic signature associated to the various syntactic and semantic categories is extended into a background algebraic specification capable of proving precisely all the equalities over ground terms (i.e., terms containing no variables). As discussed in Section 2.4, this assumption is quite reasonable, because any computational domain is isomorphic to an initial algebra over a finite algebraic specification, and that the latter can prove precisely all the ground equational properties of the domain/initial algebra. Consider, for example, the side condition $\sigma(x) \neq \perp$ of the rules (BIGSTEP-LOOKUP) and (BIGSTEP-ASGN) in Figure 3.7. Its algebraic variant is the term $\sigma(X) \neq \perp$ of *Bool* sort, where σ and X are variables of sorts *State* and *Id*, respectively. We therefore assume that any ground instance of this *Bool* term (obtained for a ground/concrete instance of the variables σ and X) can be proved using the background algebraic theory equal to either true (which means that the map is defined in the given variable) or false.

Consider now a general-purpose big-step rule of the form

$$\frac{C_1 \Downarrow R_1 \quad C_2 \Downarrow R_2 \quad \dots \quad C_n \Downarrow R_n}{C_0 \Downarrow R_0} \quad [\text{if } condition]$$

where $C_0, C_1, C_2, \dots, C_n$ are configurations holding fragments of program together with all the needed semantic components, $R_0, R_1, R_2, \dots, R_n$ are result configurations, and *condition* is some optional side condition. As one may expect, we translate it into the rewrite logic rule

$$(\forall X) \bar{C}_0 \rightarrow \bar{R}_0 \text{ if } \bar{C}_1 \rightarrow \bar{R}_1 \wedge \bar{C}_2 \rightarrow \bar{R}_2 \wedge \dots \wedge \bar{C}_n \rightarrow \bar{R}_n [\wedge \overline{condition}].$$

where X is the set of parameters, or meta-variables, that occur in the big-step proof rule (schema), now regarded as variables. Therefore, the big-step SOS rule premises and side conditions are both turned into conditions of the corresponding rewrite rule. The sequent premises become rewrites in the condition, while the side conditions become simple Boolean checks.

We make two reasonable assumptions about big-step semantics: (1) configurations cannot be nested; and (2) result configurations are irreducible.

Theorem 13. (Faithful embedding of big-step SOS into rewrite logic) *For any big-step operational semantics definition BIGSTEP, and any BIGSTEP appropriate configuration C and result configuration R , the following equivalence holds*

$$\text{BIGSTEP} \vdash C \Downarrow R \iff \mathcal{R}_{\text{BIGSTEP}} \vdash \bar{C} \rightarrow^1 \bar{R},$$

where $\mathcal{R}_{\text{BIGSTEP}}$ is the rewrite logic semantic definition obtained from BIGSTEP by translating each rule in BIGSTEP as above. (Recall from Section 2.5 that \rightarrow^1 is the one-step rewriting relation obtained by dropping the reflexivity and transitivity rules of rewrite logic. Also, since C and R are parameter-free—parameters only appear in rules—, \bar{C} and \bar{R} are ground terms.)

Proof. Before we proceed with the proof, let us understand how the assumptions about BigSTEP and the use of \rightarrow^1 affect the rewrite logic proof derivations that can be performed with $\mathcal{R}_{\text{BigSTEP}}$. First, note that the Reflexivity and Transitivity proof rules of rewrite logic (see Section 2.5) will never apply. Second, since the rules in $\mathcal{R}_{\text{BigSTEP}}$ correspond to rules in BigSTEP between configurations and there are no other rules in $\mathcal{R}_{\text{BigSTEP}}$, and since configurations cannot be nested, we conclude that the Congruence rule of rewrite logic will never apply either. Third, since the rules of BigSTEP add no equations to $\mathcal{R}_{\text{BigSTEP}}$, the equations of $\mathcal{R}_{\text{BigSTEP}}$ correspond all to the background algebraic theory used for domain reasoning; thus, the Equality rule of rewrite logic can be used for domain reasoning and for nothing else. Finally, since configurations cannot be nested and since the result configurations of BigSTEP are irreducible, we conclude that the Replacement rule of rewrite logic can only instantiate rules in $\mathcal{R}_{\text{BigSTEP}}$ by substituting their variables with terms (which can be ground or not), but it can perform no “inner” concurrent rewrites. Therefore, the capabilities of rewrite logic are significantly crippled by $\mathcal{R}_{\text{BigSTEP}}$, as the only deductions that $\mathcal{R}_{\text{BigSTEP}}$ can perform are domain (equational) reasoning and rule instantiation.

Let Σ be the signature of the assumed background algebraic formalization of configurations.

Let us first assume that $\text{BigSTEP} \vdash C \Downarrow R$ and let us prove that $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C} \rightarrow^1 \overline{R}$. We do this proof by structural induction on the BigSTEP proof/derivation tree of $C \Downarrow R$. The last proof step in the tree deriving $C \Downarrow R$ must correspond to some instance of a rule (schema) of the form

$$\frac{C_1 \Downarrow R_1 \quad \dots \quad C_m \Downarrow R_m}{C_0 \Downarrow R_0} \text{ if } condition$$

where $m \geq 0$ and where the *condition* may be missing, in which case we just assume it to be *true*. If \mathcal{X} is the set of parameters, or meta-variables, that occur in this proof rule, then the last proof step in the derivation of $C \Downarrow R$ consists of an instance of the parameters in \mathcal{X} that yields corresponding instances $(C, R, C'_1, R'_1, \dots, C'_m, R'_m, \text{true})$ of, respectively, $(C_0, R_0, C_1, R_1, \dots, C_m, R_m, \text{condition})$, such that $\text{BigSTEP} \vdash C'_i \Downarrow R'_i$ for all $1 \leq i \leq m$; moreover, domain reasoning is allowed at any moment, in particular to evaluate the instance of *condition* to *true*. By the (structural) induction hypothesis we have that $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C'_i} \rightarrow^1 \overline{R'_i}$ for all $1 \leq i \leq m$. More algebraically, the above say that there is a map $\theta : \mathcal{X} \rightarrow T_\Sigma$ such that:

- (domain reasoning) $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C} = \theta(\overline{C_0})$ and $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{R} = \theta(\overline{R_0})$;
- (domain reasoning) $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C'_i} = \theta(\overline{C_i})$ and $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{R'_i} = \theta(\overline{R_i})$ for all $1 \leq i \leq m$;
- (domain reasoning) $\mathcal{R}_{\text{BigSTEP}} \vdash \theta(\overline{\text{condition}}) = \text{true}$;
- (induction hypothesis) $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C'_i} \rightarrow^1 \overline{R'_i}$ for all $1 \leq i \leq m$.

Since $\mathcal{R}_{\text{BigSTEP}}$ contains the rule

$$(\forall \mathcal{X}) \overline{C_0} \rightarrow \overline{R_0} \text{ if } \overline{C_1} \rightarrow \overline{R_1} \wedge \overline{C_2} \rightarrow \overline{R_2} \wedge \dots \wedge \overline{C_m} \rightarrow \overline{R_m} \wedge \overline{\text{condition}},$$

the Replacement and Equality rules of rewrite logic then give us $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C} \rightarrow^1 \overline{R}$.

Let us now assume that $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C} \rightarrow^1 \overline{R}$ and prove that $\text{BigSTEP} \vdash C \Downarrow R$. Since C and R are ground configurations and since all the rewrite rules derived in any given rewrite logic proof contain the same set of variables, it follows that all the rewrite rules derived as part of the proof of $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C} \rightarrow^1 \overline{R}$ are also ground. We show by structural induction on the rewrite logic derivation tree that any derivable sequent $\mathcal{R}_{\text{BigSTEP}} \vdash u \rightarrow^1 v$ with u and v ground is of the form $\mathcal{R}_{\text{BigSTEP}} \vdash \overline{C_u} \rightarrow^1 \overline{R_v}$, where C_u is a configuration and R_v a result configuration in BigSTEP such that $\text{BigSTEP} \vdash C_u \Downarrow R_v$. There are two cases to analyze:

- The last step used the Equality proof rule of rewrite logic, that is, $\mathcal{R}_{\text{BIGSTEP}} \vdash u = u'$, $\mathcal{R}_{\text{BIGSTEP}} \vdash u' \rightarrow^1 v'$, and $\mathcal{R}_{\text{BIGSTEP}} \vdash v' = v$. Then by the induction hypothesis there is a configuration $C_{u'}$ and a result configuration $R_{v'}$ such that $u' = \overline{C_{u'}}$, $v' = \overline{R_{v'}}$ and $\text{BIGSTEP} \vdash C_{u'} \Downarrow R_{v'}$. Since $\mathcal{R}_{\text{BIGSTEP}} \vdash u = u'$ and $\mathcal{R}_{\text{BIGSTEP}} \vdash v' = v$, and since equational deduction preserves the sorts of the terms proved equal, it follows that u and v are also of configuration sort. Then let C_u and R_v be the configurations corresponding to u and v , respectively, that is, $u = \overline{C_u}$ and $v = \overline{R_v}$. Since domain properties can be tacitly used in big-step derivations, we conclude that $\text{BIGSTEP} \vdash C_u \Downarrow R_v$ and R_v is a result configuration.
- The last step is an instance of the Replacement rule of rewrite logic, with the rewrite rule

$$(\forall \mathcal{X}) \overline{C_0} \rightarrow \overline{R_0} \text{ if } \overline{C_1} \rightarrow \overline{R_1} \wedge \overline{C_2} \rightarrow \overline{R_2} \wedge \dots \wedge \overline{C_m} \rightarrow \overline{R_m} \wedge \overline{\text{condition}}$$

Then there is some ground substitution $\theta : \mathcal{X} \rightarrow T_\Sigma$ and ground terms $u_1, v_1, \dots, u_m, v_m$ such that $u = \theta(\overline{C_0})$ and $v = \theta(\overline{R_0})$, $\theta(\overline{\text{condition}}) = \text{true}$, and $u_i = \theta(\overline{C_i})$ and $v_i = \theta(\overline{R_i})$ for all $1 \leq i \leq m$. By the induction hypothesis, there are configurations C_{u_1}, \dots, C_{u_m} and result configurations R_{v_1}, \dots, R_{v_m} such that $u_i = \overline{C_{u_i}}$, $v_i = \overline{R_{v_i}}$, and $\text{BIGSTEP} \vdash C_{u_i} \Downarrow R_{v_i}$ for all $1 \leq i \leq m$. Let C_u and R_v be the configurations corresponding to u and v , respectively, that is, $u = \overline{C_u}$ and $v = \overline{R_v}$. We have thus found instances $(C_u, R_u, C_{u_1}, R_{v_1}, \dots, C_{u_m}, R_{v_m}, \text{true})$ of $(C_0, R_0, C_1, R_1, \dots, C_m, R_m, \text{condition})$, such that $\text{BIGSTEP} \vdash C_{u_i} \Downarrow R_{v_i}$ for all $1 \leq i \leq m$. In other words, we constructed an instance of the BIGSTEP rule

$$\frac{C_1 \Downarrow R_1 \quad \dots \quad C_m \Downarrow R_m}{C_0 \Downarrow R_0} \text{ if } \text{condition}$$

with derivations for all its premises. Hence, $\text{BIGSTEP} \vdash C \Downarrow R$.

□

The non-nestedness assumption on configurations in BIGSTEP guarantees that the resulting rewrite rules in $\mathcal{R}_{\text{BIGSTEP}}$ only apply at the top of the term they rewrite. The irreducibility of the result configurations guarantees that $\mathcal{R}_{\text{BIGSTEP}}$ does not do more rewrite steps than intended, because rewriting is an inherently transitively closed relation, while the big-step relation \Downarrow is not.

Since one typically perceives parameters as variables anyway, the only apparent difference between BIGSTEP and $\mathcal{R}_{\text{BIGSTEP}}$ is the different notational conventions they use (\rightarrow instead of \Downarrow and conditional rewrite rules instead of conditional deduction rules). As Theorem 13 shows, there is a one-to-one correspondence also between their corresponding “computations” (or executions, or derivations). Therefore, $\mathcal{R}_{\text{BIGSTEP}}$ is the big-step operational semantics BIGSTEP, and *not* an encoding of it.

At our knowledge, there is no rewrite engine³ that supports the one-step rewrite relation \rightarrow^1 (that appears in Theorem 13). Indeed, rewrite engines aim at high-performance implementations of the general rewrite relation \rightarrow , which may even involve parallel rewriting (see Section 2.5 for the precise definitions of \rightarrow^1 and \rightarrow); \rightarrow^1 is meaningful only from a theoretical perspective and there is little to no practical motivation for an efficient implementation of it. Therefore, in order to execute the rewrite theory $\mathcal{R}_{\text{BIGSTEP}}$ resulting from the mechanical translation of big-step semantics BIGSTEP, one needs to take some precautions to ensure that \rightarrow^1 is actually identical to \rightarrow .

Given any rewrite theory \mathcal{R} , a sufficient condition for \rightarrow^1 to be the same as \rightarrow in \mathcal{R} is for the right-hand-sides of the rules in \mathcal{R} to generate terms which make any context that contains them unmatchable by any rule

³Maude’s `rewrite[1]` command does not inhibit the transitive closure of the rewrite relation, it only stops the rewrite engine on a given term after *one* application of a rule on that term; however, many (transitive) applications of rules are allowed when solving the condition of that rule.

in \mathcal{R} . Fortunately, the two assumptions we made about our original BIGSTEP semantics guarantee this property. First, the non-nestedness assumption guarantees that configurations can only appear at the top of a term, so the only contexts that contain configurations are the configurations themselves. Second, the irreducibility of result configurations assumption ensures that the rules in $\mathcal{R}_{\text{BIGSTEP}}$ will never match any result configurations. Therefore, we can conclude the following important

Corollary 3. *Under the same hypotheses and assumptions as in Theorem 13,*

$$\text{BIGSTEP} \vdash C \Downarrow R \iff \mathcal{R}_{\text{BIGSTEP}} \vdash \bar{C} \rightarrow^1 \bar{R} \iff \mathcal{R}_{\text{BIGSTEP}} \vdash \bar{C} \rightarrow \bar{R}.$$

Our current $\text{BIGSTEP}(\text{IMP})$ semantics verifies our second assumption, since the configurations to the left of \Downarrow and the result configurations to the right of \Downarrow are always distinct. Unfortunately, in general that may not always be the case. For example, when we extend IMP with side effects in Section 3.10, the state also needs to be part of result configurations, so the semantics of integers is going to be given by an unconditional rule of the form $\langle i, \sigma \rangle \Downarrow \langle i, \sigma \rangle$, which after translation becomes the rewrite rule $\langle i, \sigma \rangle \rightarrow \langle i, \sigma \rangle$. This rule will make the rewrite relation \rightarrow not terminate anymore (although the relation \rightarrow^1 will still terminate). There are at least two simple ways to ensure the irreducibility of result configurations, and thus make Corollary 3 still hold:

1. It is highly expected that the only big-step rules in BIGSTEP having a result configuration to the left of \Downarrow are unconditional rules of the form $R \Downarrow R$; such rules typically say that a value is already evaluated. If that is the case, then one can simply drop all the corresponding rules $\bar{R} \rightarrow \bar{R}$ from $\mathcal{R}_{\text{BIGSTEP}}$ and the resulting rewrite theory, say $\mathcal{R}'_{\text{BIGSTEP}}$ still has the property $\text{BIGSTEP} \vdash C \Downarrow R \iff \mathcal{R}'_{\text{BIGSTEP}} \vdash \bar{C} \rightarrow \bar{R}$, which is desirable in order to execute the big-step definition on rewrite engines, although the property $\text{BIGSTEP} \vdash C \Downarrow R \iff \mathcal{R}'_{\text{BIGSTEP}} \vdash \bar{C} \rightarrow^1 \bar{R}$ will not hold anymore, because, e.g., even though $R \Downarrow R$ is a rule in BIGSTEP , it is not the case that $\mathcal{R}'_{\text{BIGSTEP}} \vdash \bar{R} \rightarrow^1 \bar{R}$.
2. If BIGSTEP contains pairs $R' \Downarrow R$ where R' and R are possibly different result configurations, then one can apply the following general procedure. Change or augment the syntax of the configurations to the left or to the right of \Downarrow , so that those changed or augmented configurations will always be different from the other ones. This is the technique employed in our representation of small-step operational semantics in rewriting logic in Section 3.3. More precisely, we prepend all the configurations to the left of the rewrite relation in $\mathcal{R}_{\text{BIGSTEP}}$ with a circle \circ , e.g., $\circ C \rightarrow R$, with the intuition that the circled configurations are *active*, while the other ones are *inactive*.

Regardless of how the desired property $\text{BIGSTEP} \vdash C \Downarrow R \iff \mathcal{R}_{\text{BIGSTEP}} \vdash \bar{C} \rightarrow \bar{R}$ is ensured, note that, unfortunately, $\mathcal{R}_{\text{BIGSTEP}}$ lacks the main strengths of rewrite logic that make it a good formalism for concurrency: in rewrite logic, rewrite rules can apply under any context and in parallel. Indeed, the rules of $\mathcal{R}_{\text{BIGSTEP}}$ can only apply at the top, sequentially.

Big-Step SOS of IMP in Rewrite Logic

Figure 3.8 shows the rewrite logic theory $\mathcal{R}_{\text{BIGSTEP}(\text{IMP})}$ that is obtained by applying the procedure above to the big-step semantics of IMP , $\text{BIGSTEP}(\text{IMP})$, in Figure 3.7. We have used the rewrite logic convention that variables start with upper-case letters. For the state variable, we used σ , that is, a larger σ symbol.

Note how the three side conditions that appear in the proof system in Figure 3.7 turned into normal conditions of rewrite rules. In particular, the two side conditions saying that $\sigma(x)$ is defined became the algebraic term $\sigma(X) \neq \perp$ of Boolean sort.

The following corollary of Theorem 13 and Corollary 3 establishes the faithfulness of the representation of the big-step operational semantics of IMP in rewrite logic:

$$\begin{array}{ll}
\langle I, \sigma \rangle \rightarrow \langle I \rangle & \\
\langle X, \sigma \rangle \rightarrow \langle \sigma(X) \rangle & \text{if } \sigma(X) \neq \perp \\
\langle A_1 + A_2, \sigma \rangle \rightarrow \langle I_1 +_{int} I_2 \rangle & \text{if } \langle A_1, \sigma \rangle \rightarrow \langle I_1 \rangle \wedge \langle A_2, \sigma \rangle \rightarrow \langle I_2 \rangle \\
\langle A_1 / A_2, \sigma \rangle \rightarrow \langle I_1 /_{int} I_2 \rangle & \text{if } \langle A_1, \sigma \rangle \rightarrow \langle I_1 \rangle \wedge \langle A_2, \sigma \rangle \rightarrow \langle I_2 \rangle \wedge I_2 \neq 0 \\
\langle A_1 \leq A_2, \sigma \rangle \rightarrow \langle I_1 \leq_{int} I_2 \rangle & \text{if } \langle A_1, \sigma \rangle \rightarrow \langle I_1 \rangle \wedge \langle A_2, \sigma \rangle \rightarrow \langle I_2 \rangle \\
\langle T, \sigma \rangle \rightarrow \langle T \rangle & \\
\langle ! B, \sigma \rangle \rightarrow \langle \text{false} \rangle & \text{if } \langle B, \sigma \rangle \rightarrow \langle \text{true} \rangle \\
\langle ! B, \sigma \rangle \rightarrow \langle \text{true} \rangle & \text{if } \langle B, \sigma \rangle \rightarrow \langle \text{false} \rangle \\
\langle B_1 \ \&\& \ B_2, \sigma \rangle \rightarrow \langle \text{false} \rangle & \text{if } \langle B_1, \sigma \rangle \rightarrow \langle \text{false} \rangle \\
\langle B_1 \ \&\& \ B_2, \sigma \rangle \rightarrow \langle T \rangle & \text{if } \langle B_1, \sigma \rangle \rightarrow \langle \text{true} \rangle \wedge \langle B_2, \sigma \rangle \rightarrow \langle T \rangle \\
\langle \{\}, \sigma \rangle \rightarrow \langle \sigma \rangle & \\
\langle \{ S \}, \sigma \rangle \rightarrow \langle \sigma' \rangle & \text{if } \langle S, \sigma \rangle \rightarrow \langle \sigma' \rangle \\
\langle X = A; , \sigma \rangle \rightarrow \langle \sigma[I/X] \rangle & \text{if } \langle A, \sigma \rangle \rightarrow \langle I \rangle \wedge \sigma(X) \neq \perp \\
\langle S_1 \ S_2, \sigma \rangle \rightarrow \langle \sigma_2 \rangle & \text{if } \langle S_1, \sigma \rangle \rightarrow \langle \sigma_1 \rangle \wedge \langle S_2, \sigma_1 \rangle \rightarrow \langle \sigma_2 \rangle \\
\langle \text{if } (B) \ S_1 \ \text{else} \ S_2, \sigma \rangle \rightarrow \langle \sigma_1 \rangle & \text{if } \langle B, \sigma \rangle \rightarrow \langle \text{true} \rangle \wedge \langle S_1, \sigma \rangle \rightarrow \langle \sigma_1 \rangle \\
\langle \text{if } (B) \ S_1 \ \text{else} \ S_2, \sigma \rangle \rightarrow \langle \sigma_2 \rangle & \text{if } \langle B, \sigma \rangle \rightarrow \langle \text{false} \rangle \wedge \langle S_2, \sigma \rangle \rightarrow \langle \sigma_2 \rangle \\
\langle \text{while } (B) \ S, \sigma \rangle \rightarrow \langle \sigma \rangle & \text{if } \langle B, \sigma \rangle \rightarrow \langle \text{false} \rangle \\
\langle \text{while } (B) \ S, \sigma \rangle \rightarrow \langle \sigma' \rangle & \text{if } \langle B, \sigma \rangle \rightarrow \langle \text{true} \rangle \wedge \langle \{ S \ \text{while } (B) \ S \}, \sigma \rangle \rightarrow \langle \sigma' \rangle \\
\langle \text{int } X!; S \rangle \rightarrow \langle \sigma \rangle & \text{if } \langle S, X! \mapsto 0 \rangle \rightarrow \langle \sigma \rangle
\end{array}$$

Figure 3.8: $\mathcal{R}_{\text{BIGSTEP}(\text{IMP})}$: the big-step SOS of IMP in rewrite logic.

Corollary 4. $\text{BIGSTEP}(\text{IMP}) \vdash C \Downarrow R \iff \mathcal{R}_{\text{BIGSTEP}(\text{IMP})} \vdash \overline{C} \rightarrow \overline{R}$.

Therefore, there is no perceivable computational difference between the IMP-specific proof system $\text{BIGSTEP}(\text{IMP})$ and generic rewrite logic deduction using the IMP-specific rewrite rules in $\mathcal{R}_{\text{BIGSTEP}(\text{IMP})}$, so the two are faithfully equivalent.

★ Maude Definition of IMP Big-Step SOS

Figure 3.9 shows a straightforward Maude representation of the rewrite theory $\mathcal{R}_{\text{BIGSTEP}(\text{IMP})}$ in Figure 3.8, including a representation of the algebraic signature in Figure 3.6 for configurations as needed for big-step SOS. The Maude module `IMP-SEMANTICS-BIGSTEP` in Figure 3.9 is executable, so Maude, through its rewriting capabilities, yields an interpreter for IMP; for example, the command

```
rewrite < sumPgm > .
```

where `sumPgm` is the first program defined in the module `IMP-PROGRAMS` in Figure 3.4, produces a result of the form (the exact statistics are irrelevant here, so they were replaced by “...”):

```
rewrites: 5218 in ... cpu (... real) (... rewrites/second)
result Configuration: < n |-> 0 , s |-> 5050 >
```

The obtained IMP interpreter actually has acceptable performance; for example, all the programs in Figure 3.4 together take a fraction of a second to execute on conventional PCs or laptops.

In fact, Maude needs only one rewrite logic step to rewrite any configuration; in particular,

```
rewrite [1] < sumPgm > .
```

```

mod IMP-CONFIGURATIONS-BIGSTEP is including IMP-SYNTAX + STATE .
  sort Configuration .
  op <_,_> : AExp State -> Configuration .
  op <_> : Int -> Configuration .
  op <_,_> : BExp State -> Configuration .
  op <_> : Bool -> Configuration .
  op <_,_> : Stmt State -> Configuration .
  op <_> : State -> Configuration .
  op <_> : Pgm -> Configuration .
endm

mod IMP-SEMANTICS-BIGSTEP is including IMP-CONFIGURATIONS-BIGSTEP .
  var X : Id . var Xl : List{Id} . var Sigma Sigma' Sigma1 Sigma2 : State .
  var I I1 I2 : Int . var T : Bool .
  var A A1 A2 : AExp . var B B1 B2 : BExp . var S S1 S2 : Stmt .

  rl < I,Sigma > => < I > .
  crl < X,Sigma > => < Sigma(X) >
  if Sigma(X) /=Bool undefined .
  crl < A1 + A2,Sigma > => < I1 +Int I2 >
  if < A1,Sigma > => < I1 > /\ < A2,Sigma > => < I2 > .
  crl < A1 / A2,Sigma > => < I1 /Int I2 >
  if < A1,Sigma > => < I1 > /\ < A2,Sigma > => < I2 > /\ I2 /=Bool 0 .

  rl < T,Sigma > => < T > .
  crl < A1 <= A2,Sigma > => < I1 <=Int I2 >
  if < A1,Sigma > => < I1 > /\ < A2,Sigma > => < I2 > .
  crl < ! B,Sigma > => < false >
  if < B,Sigma > => < true > .
  crl < ! B,Sigma > => < true >
  if < B,Sigma > => < false > .
  crl < B1 && B2,Sigma > => < false >
  if < B1,Sigma > => < false > .
  crl < B1 && B2,Sigma > => < T >
  if < B1,Sigma > => < true > /\ < B2,Sigma > => < T > .

  rl < {},Sigma > => < Sigma > .
  crl < {S},Sigma > => < Sigma' >
  if < S,Sigma > => < Sigma' > .
  crl < X = A ;,Sigma > => < Sigma[I / X] >
  if < A,Sigma > => < I > /\ Sigma(X) /=Bool undefined .
  crl < S1 S2,Sigma > => < Sigma2 >
  if < S1,Sigma > => < Sigma1 > /\ < S2,Sigma1 > => < Sigma2 > .
  crl < if (B) S1 else S2,Sigma > => < Sigma1 >
  if < B,Sigma > => < true > /\ < S1,Sigma > => < Sigma1 > .
  crl < if (B) S1 else S2,Sigma > => < Sigma2 >
  if < B,Sigma > => < false > /\ < S2,Sigma > => < Sigma2 > .
  crl < while (B) S,Sigma > => < Sigma >
  if < B,Sigma > => < false > .
  crl < while (B) S,Sigma > => < Sigma' >
  if < B,Sigma > => < true > /\ < S while (B) S,Sigma > => < Sigma' > .

  crl < int Xl ; S > => < Sigma >
  if < S,(Xl |-> 0) > => < Sigma > .
endm

```

Figure 3.9: The big-step SOS of IMP in Maude, including the definition of configurations.

will give the same output as above. Recall from Section 2.5.6 that Maude performs a potentially exhaustive search to satisfy the rewrites in rule conditions. Thus, a large number of rule instances can be attempted in order to apply one conditional rule, so a `rewrite [1]` command can take a long time; it may not even terminate. Nevertheless, thanks to Theorem 13, Maude’s implicit search mechanism in conditions effectively achieves a proof searcher for big-step SOS derivations.

Once one has a rewrite logic definition in Maude, one can use any of the general-purpose tools provided by Maude on that definition; the rewrite engine is only one of them. For example, one can exhaustively search for all possible behaviors of a program using the `search` command:

```
search < sumPgm > =>! Cfg:Configuration .
```

Since our IMP language so far is deterministic, the `search` command will not discover any new behaviors. In fact, the search command will only discover two configurations in total, the original configuration `< sumPgm >` and the result configuration `< n |-> 0 & s |-> 5050 >`. However, as shown in Section 3.5 where we extend IMP with various language features, the `search` command can indeed show all the behaviors of a non-deterministic program (restricted only by the limitations of the particular semantic style employed).

3.2.4 Defining a Type System for IMP Using Big-Step SOS

Big-step SOS is routinely used to define type systems for programming languages, even though in most cases this connection is not made explicit. In this section we demonstrate the use of big-step SOS for defining a type system for IMP, following the same steps as above but more succinctly. Type systems is a broad subject, with many variations and important applications to programming languages. Our intention in this section is twofold: on the one hand we show that big-step SOS is not limited to only defining language semantics, and, on the other hand, we introduce the reader to type systems by means of a very simple example.

The idea underlying big-step SOS definitions of type systems is that a given program or fragment of program in a given type environment reduces, in one big step, to its type. Like states, type environments are also partial mappings, but from variable names into types instead of values. A common notation for a type judgment is $\Gamma \vdash c : \tau$, where Γ is a type environment, c is a program or fragment, and τ is a type. This type judgment reads “in type environment Γ , program or fragment c has type τ ”. One can find countless variations of the notation for type judgments in the literature, usually adding more items (pieces of information) to the left of \vdash , to its right, or as subscripts or superscripts of it. There is, unfortunately, no well-established notation for all type judgments. Nevertheless, type judgments are special big-step sequents relating two special configurations, one including the givens and the other the results. For example, a simple type judgment $\Gamma \vdash c : \tau$ like above can be regarded as a big-step sequent $\langle c, \Gamma \rangle \Downarrow \langle \tau \rangle$. However, this notation is not preferred.

Figure 3.10 depicts our type system for IMP, which is a nothing but a big-step SOS proof system. We, however, follow the more conventional notation for type judgments discussed above, with one slight change: since in IMP variables are intended to hold only integer values, there is no need for type environments; instead, we replace them by lists of variables, each meant to have the type `int`. Therefore, $x/\vdash c : \tau$ with c and τ as above but with $x/$ a list of variables reads as follows: “when the variables in $x/$ are defined, c has the type τ ”. We drop the list of variables from the typing judgments of programs, because that would be empty anyway. The big-step SOS rules in Figure 3.10 define the typing policy of each language construct of IMP, guaranteeing all together that a program p types, that is, that $\vdash p : \text{pgm}$ is derivable if and only if each construct is used according to its intended typing policy and, moreover, that p declares each variable that it uses. For our simple IMP language, a CFG parser using the syntax defined in Figure 3.1 would already guarantee that each construct is used as intended. Note, however, that the second desired property of our type system (each used variable is declared) is context dependent.

$xl \vdash i : \text{int}$	(BIGSTEPTypeSystem-INT)
$xl \vdash x : \text{int} \quad \text{if } x \in xl$	(BIGSTEPTypeSystem-LOOKUP)
$\frac{xl \vdash a_1 : \text{int} \quad xl \vdash a_2 : \text{int}}{xl \vdash a_1 + a_2 : \text{int}}$	(BIGSTEPTypeSystem-ADD)
$\frac{xl \vdash a_1 : \text{int} \quad xl \vdash a_2 : \text{int}}{xl \vdash a_1 / a_2 : \text{int}}$	(BIGSTEPTypeSystem-DIV)
$xl \vdash t : \text{bool} \quad \text{if } t \in \{\text{true}, \text{false}\}$	(BIGSTEPTypeSystem-BOOL)
$\frac{xl \vdash a_1 : \text{int} \quad xl \vdash a_2 : \text{int}}{xl \vdash a_1 \leq a_2 : \text{bool}}$	(BIGSTEPTypeSystem-LEQ)
$\frac{xl \vdash b : \text{bool}}{xl \vdash ! b : \text{bool}}$	(BIGSTEPTypeSystem-NOT)
$\frac{xl \vdash b_1 : \text{bool} \quad xl \vdash b_2 : \text{bool}}{xl \vdash b_1 \ \&\& \ b_2 : \text{bool}}$	(BIGSTEPTypeSystem-AND)
$xl \vdash \{\} : \text{block}$	(BIGSTEPTypeSystem-EMPTY-BLOCK)
$\frac{xl \vdash s : \tau}{xl \vdash \{s\} : \text{block}} \quad \text{if } \tau \in \{\text{block}, \text{stmt}\}$	(BIGSTEPTypeSystem-BLOCK)
$\frac{xl \vdash a : \text{int}}{xl \vdash x = a ; \text{stmt}} \quad \text{if } x \in xl$	(BIGSTEPTypeSystem-ASGN)
$\frac{xl \vdash s_1 : \tau_1 \quad xl \vdash s_2 : \tau_2}{xl \vdash s_1 \ s_2 : \text{stmt}} \quad \text{if } \tau_1, \tau_2 \in \{\text{block}, \text{stmt}\}$	(BIGSTEPTypeSystem-SEQ)
$\frac{xl \vdash b : \text{bool} \quad xl \vdash s_1 : \text{block} \quad xl \vdash s_2 : \text{block}}{xl \vdash \text{if } (b) \ s_1 \ \text{else } s_2 : \text{stmt}}$	(BIGSTEPTypeSystem-IF)
$\frac{xl \vdash b : \text{bool} \quad xl \vdash s : \text{block}}{xl \vdash \text{while } (b) \ s : \text{stmt}}$	(BIGSTEPTypeSystem-WHILE)
$\frac{xl \vdash s : \tau}{\vdash \text{int } xl ; s : \text{pgm}} \quad \text{if } \tau \in \{\text{block}, \text{stmt}\}$	(BIGSTEPTypeSystem-PGM)

Figure 3.10: BIGSTEPTypeSystem(IMP) — Type system of IMP using big-step SOS ($xl \in \mathbf{List}\{Id\}$; $i \in Int$; $x \in Id$; $a, a_1, a_2 \in AExp$; $b, b_1, b_2 \in BExp$; $s, s_1, s_2 \in Stmt$).

Let us next use the type system in Figure 3.10 to type the program `sumPgm` in Figure 3.4. We split the proof tree in proof subtrees. Note first that using the rules (BIGSTEPTypeSYSTEM-INT) (first level), (BIGSTEPTypeSYSTEM-ASGN) (second level) and (BIGSTEPTypeSYSTEM-SEQ) (third level), we can derive the following proof tree, say $tree_1$:

$$tree_1 = \left\{ \frac{\frac{\cdot}{n, s \vdash 100 : \text{int}} \quad \frac{\cdot}{n, s \vdash 0 : \text{int}}}{\frac{n, s \vdash (n = 100;) : \text{stmt} \quad n, s \vdash (s = 0;) : \text{stmt}}{n, s \vdash (n = 100; s = 0;) : \text{stmt}}} \right.$$

Similarly, using rules (BIGSTEPTypeSYSTEM-LOOKUP) and (BIGSTEPTypeSYSTEM-INT) (first level), (BIGSTEPTypeSYSTEM-LEQ) (second level), and (BIGSTEPTypeSYSTEM-NOT) (third level), we can derive the following proof tree, say $tree_2$:

$$tree_2 = \left\{ \frac{\frac{\cdot}{n, s \vdash n : \text{int}} \quad \frac{\cdot}{n, s \vdash 0 : \text{int}}}{\frac{n, s \vdash (n \leq 0) : \text{bool}}{n, s \vdash (! (n \leq 0)) : \text{bool}}} \right.$$

Similarly, we can derive the following proof tree, say $tree_3$:

$$tree_3 = \left\{ \frac{\frac{\frac{\cdot}{n, s \vdash s : \text{int}} \quad \frac{\cdot}{n, s \vdash n : \text{int}}}{n, s \vdash (s + n) : \text{int}} \quad \frac{\frac{\cdot}{n, s \vdash n : \text{int}} \quad \frac{\cdot}{n, s \vdash -1 : \text{int}}}{n, s \vdash (n + -1) : \text{int}}}{\frac{n, s \vdash (s = s + n;) : \text{stmt} \quad n, s \vdash (n = n + -1;) : \text{stmt}}{n, s \vdash (s = s + n; n = n + -1;) : \text{stmt}}}$$

$$\frac{n, s \vdash (s = s + n; n = n + -1;) : \text{stmt}}{n, s \vdash \{ s = s + n; n = n + -1; \} : \text{block}}$$

Finally, we can now derive the tree that proves that `sumPgm` is well-typed:

$$\frac{\frac{tree_1 \quad \frac{tree_2 \quad tree_3}{n, s \vdash (\text{while } (! (n \leq 0)) \{ s = s + n; n = n + -1; \}) : \text{stmt}}}{n, s \vdash (n = 100; s = 0; \text{while } (! (n \leq 0)) \{ s = s + n; n = n + -1; \}) : \text{stmt}}}{\vdash (\text{int } n, s; n = 100; s = 0; \text{while } (! (n \leq 0)) \{ s = s + n; n = n + -1; \}) : \text{pgm}}$$

A major role of a type system is to filter out a set of programs which are obviously wrong. Unfortunately, it is impossible to filter out precisely those programs which would execute erroneously. For example, note that a division is considered type safe whenever its two arguments type to integers, but no check is being made on whether the denominator is 0 or not. Indeed, statically checking whether an expression has a certain value at a certain point in a program is an undecidable problem. Also, no check is being made on whether a detected type error is reachable or not (if unreachable, the detected type error will never show up at runtime). Statically checking whether a certain point in a program is reachable is also an undecidable problem. One should therefore be aware of the fact that in general a type system may allow programs which run into errors when executed and, moreover, that it may reject programs which would execute correctly.

$$\begin{aligned}
& \langle I, Xl \rangle \rightarrow \langle \text{int} \rangle \\
& \langle X, (Xl, X, Xl') \rangle \rightarrow \langle \text{int} \rangle \\
& \langle A_1 + A_2, Xl \rangle \rightarrow \langle \text{int} \rangle \text{ if } \langle A_1, Xl \rangle \rightarrow \langle \text{int} \rangle \wedge \langle A_2, Xl \rangle \rightarrow \langle \text{int} \rangle \\
& \langle \div IMP A_1 A_2, Xl \rangle \rightarrow \langle \text{int} \rangle \text{ if } \langle A_1, Xl \rangle \rightarrow \langle \text{int} \rangle \wedge \langle A_2, Xl \rangle \rightarrow \langle \text{int} \rangle \\
& \langle A_1 \leq A_2, Xl \rangle \rightarrow \langle \text{bool} \rangle \text{ if } \langle A_1, Xl \rangle \rightarrow \langle \text{int} \rangle \wedge \langle A_2, Xl \rangle \rightarrow \langle \text{int} \rangle \\
& \langle T, Xl \rangle \rightarrow \langle \text{bool} \rangle \\
& \langle ! B, Xl \rangle \rightarrow \langle \text{bool} \rangle \text{ if } \langle B, Xl \rangle \rightarrow \langle \text{bool} \rangle \\
& \langle B_1 \&\& B_2, Xl \rangle \rightarrow \langle \text{bool} \rangle \text{ if } \langle B_1, Xl \rangle \rightarrow \langle \text{bool} \rangle \wedge \langle B_2, Xl \rangle \rightarrow \langle \text{bool} \rangle \\
& \langle \{\}, Xl \rangle \rightarrow \langle \text{block} \rangle \\
& \langle \{ S \}, Xl \rangle \rightarrow \langle \text{block} \rangle \text{ if } \langle S, Xl \rangle \rightarrow \langle ST \rangle \\
& \langle X = A ;, (Xl, X, Xl') \rangle \rightarrow \langle \text{stmt} \rangle \text{ if } \langle A, (Xl, X, Xl') \rangle \rightarrow \langle \text{int} \rangle \\
& \langle S_1 S_2, Xl \rangle \rightarrow \langle \text{stmt} \rangle \text{ if } \langle S_1, Xl \rangle \rightarrow \langle ST_1 \rangle \wedge \langle S_2, Xl \rangle \rightarrow \langle ST_2 \rangle \\
& \langle \text{if } (B) S_1 \text{ else } S_2, Xl \rangle \rightarrow \langle \text{stmt} \rangle \text{ if } \langle B, Xl \rangle \rightarrow \langle \text{bool} \rangle \wedge \langle S_1, Xl \rangle \rightarrow \langle \text{block} \rangle \wedge \langle S_2, Xl \rangle \rightarrow \langle \text{block} \rangle \\
& \langle \text{while } (B) S, Xl \rangle \rightarrow \langle \text{stmt} \rangle \text{ if } \langle B, Xl \rangle \rightarrow \langle \text{bool} \rangle \wedge \langle S, Xl \rangle \rightarrow \langle \text{block} \rangle \\
& \langle \text{int } Xl ; S \rangle \rightarrow \langle \text{pgm} \rangle \text{ if } \langle S, Xl \rangle \rightarrow \langle ST \rangle
\end{aligned}$$

Figure 3.11: $\mathcal{R}_{\text{BIGSTEPTypeSYSTEM}(\text{IMP})}$: type system of IMP using big-step SOS in rewrite logic. Assume a sort *Type* with constants `int`, `bool`, and `pgm`, and a subsort of it *StmtType* with constants `block` and `stmt`. The sorts of the involved variables is understood (from Figure 3.10), except for T which has the sort *Bool* and for ST , ST_1 and ST_2 which have the sort *StmtType*.

Figure 3.11 shows a translation of the big-step SOS in Figure 3.10 into a rewrite logic theory, following the general technique described in Section 3.2.3 but with two simple optimizations explained shortly. This translation is based on the reinterpretation of type judgments as big-step SOS sequents mentioned above. The following configurations are used in the rewrite theory in Figure 3.11:

- $\langle A, Xl \rangle$ grouping arithmetic expressions A and variable lists Xl ;
- $\langle B, Xl \rangle$ grouping Boolean expressions B and variable lists Xl ;
- $\langle S, Xl \rangle$ grouping statements S and variable lists Xl ;
- $\langle P \rangle$ holding programs P ;
- $\langle \tau \rangle$ holding types τ , which can be `int`, `bool`, `block`, `stmt`, or `pgm`.

The two optimizations refer to how to test the various memberships that occur in some of the rules' side conditions. One optimization is to use associative matching to check membership to a list; this applies to the rules (`BIGSTEPTypeSYSTEM-LOOKUP`) and (`BIGSTEPTypeSYSTEM-ASGN`). Another optimization is to use a (sub)sort as an umbrella for a set of terms of interest (constants in our case); for example, we let T be a variable of sort *Bool* (i.e., T can match both `true` and `false`) when defining the rewrite rule corresponding to (`BIGSTEPTypeSYSTEM-BOOL`), and we let ST , ST_1 , and ST_2 be variables ranging over a new subsort *StmtType* of *Type* when defining the rewrite rules corresponding to (`BIGSTEPTypeSYSTEM-BLOCK`), (`BIGSTEPTypeSYSTEM-SEQ`) and (`BIGSTEPTypeSYSTEM-PGM`).

By Corollary 3 we have that a program p is well-typed, that is, $\vdash p : \text{pgm}$ is derivable with the proof system in Figure 3.10, if and only if $\mathcal{R}_{\text{BIGSTEPTypeSYSTEM}(\text{IMP})} \vdash \langle \bar{p} \rangle \rightarrow \langle \text{pgm} \rangle$.

```

mod IMP-TYPES is
  sorts StmtType Type .  subsort StmtType < Type .
  ops block stmt : -> StmtType .
  ops int bool pgm : -> Type .
endm

mod IMP-TYPE-SYSTEM-CONFIGURATIONS-BIGSTEP is including IMP-SYNTAX + IMP-TYPES .
  sort Configuration .
  op <_,_> : AExp List{Id} -> Configuration .
  op <_,_> : BExp List{Id} -> Configuration .
  op <_,_> : Stmt List{Id} -> Configuration .
  op <_> : Pgm -> Configuration .
  op <_> : Type -> Configuration .
endm

mod IMP-TYPE-SYSTEM-BIGSTEP is including IMP-TYPE-SYSTEM-CONFIGURATIONS-BIGSTEP .
  var X : Id .  var X1 X1' : List{Id} .  var I : Int .  var T : Bool .
  var A A1 A2 : AExp .  var B B1 B2 : BExp .  var S S1 S2 : Stmt .
  var ST ST1 ST2 : StmtType .

  rl < I,X1 > => < int > .
  rl < X,(X1,X,X1') > => < int > .
  crl < A1 + A2,X1 > => < int >
    if < A1,X1 > => < int > /\ < A2,X1 > => < int > .
  crl < A1 / A2,X1 > => < int >
    if < A1,X1 > => < int > /\ < A2,X1 > => < int > .

  rl < T,X1 > => < bool > .
  crl < A1 <= A2,X1 > => < bool >
    if < A1,X1 > => < int > /\ < A2,X1 > => < int > .
  crl < ! B,X1 > => < bool >
    if < B,X1 > => < bool > .
  crl < B1 && B2,X1 > => < bool >
    if < B1,X1 > => < bool > /\ < B2,X1 > => < bool > .

  rl < {},X1 > => < block > .
  crl < {S},X1 > => < block >
    if < S,X1 > => < ST > .
  crl < X = A ;,(X1,X,X1') > => < stmt >
    if < A,(X1,X,X1') > => < int > .
  crl < S1 S2,X1 > => < stmt >
    if < S1,X1 > => < ST1 > /\ < S2,X1 > => < ST2 > .
  crl < if (B) S1 else S2,X1 > => < stmt >
    if < B,X1 > => < bool > /\ < S1,X1 > => < block > /\ < S2,X1 > => < block > .
  crl < while (B) S,X1 > => < stmt >
    if < B,X1 > => < bool > /\ < S,X1 > => < block > .

  crl < int X1 ; S > => < pgm >
    if < S,X1 > => < ST > .
endm

```

Figure 3.12: The type-system of IMP using big-step SOS in Maude, including the definition of types and configurations.

★ Maude Definition of a Type System for IMP using Big-Step SOS

Figure 3.12 shows the Maude representation of the rewrite theory $\mathcal{R}_{\text{BIGSTEPTypeSystem(IMP)}}$ in Figure 3.11, including a representation of the algebraic signature for the needed configurations. The Maude module `IMP-TYPE-SYSTEM-BIGSTEP` in Figure 3.12 is executable, so Maude, through its rewriting capabilities, yields a type checker for IMP; for example, the command

```
rewrite < sumPgm > .
```

where `sumPgm` is the first program defined in the module `IMP-PROGRAMS` in Figure 3.4, produces a result of the form (the exact statistics are irrelevant here, so they were replaced by “...”):

```
rewrites: 20 in ... cpu (... real) (... rewrites/second)
result Configuration: < pgm >
```

A type system is generally expected to be deterministic. Nevertheless, implementations of it (particularly rewrite-based ones) may mistakenly be non-deterministic (non-confluent; see Section 2.1.4). To gain confidence in the determinism of the Maude definition in Figure 3.12, one may exhaustively search for all possible behaviors yielded by the type checker:

```
search < sumPgm > =>! Cfg:Configuration .
```

As expected, this finds only one solution. This Maude definition of IMP’s type checker is very simple and one can easily see that it is confluent (it is orthogonal—see Section 2.1.4), so the search is redundant. However, the search command may be useful for testing more complex type systems.

3.2.5 Notes

Big-step structural operational semantics (big-step SOS) was introduced under the name *natural semantics* by Kahn [34] in 1987. Even though he introduced it in the limited context of defining Mini-ML, a simple pure (no side effects) version of the ML language, Kahn’s aim was to propose natural semantics as a “unified manner to present different aspects of the semantics of programming languages, such as dynamic semantics, static semantics and translation” (Section 1.1 in [34]). Kahn’s original notation for big-step sequents was $\sigma \vdash a \Rightarrow i$, with the meaning that a evaluates to i in state (or environment) σ . Kahn, like many others after him (including ourselves; e.g., Section 3.2.4), took the freedom to using a different notation for type judgments, namely $\Gamma \vdash c : \tau$, where Γ is a type environment, c is a program or fragment of program, and τ is a type. This colon notation for type judgments was already established in 1987; however, Kahn noticed that the way type systems were defined was a special instance of a more general schema, which he called natural semantics (and which is called big-step SOS here and in many other places). Big-step SOS is very natural when defining pure, deterministic and structured languages, so it quickly became very popular. However, Kahn’s terminology for “natural” semantics was inspired from its reminiscence to “natural deduction” in mathematical logic, not necessarily from the fact that it is natural to use. He demonstrated how one can use big-step SOS to define all these in his seminal paper [34], using the Mini-ML language.

As Kahn himself acknowledged, the idea of using proof systems to capture the operational semantics of programming languages goes back to Plotkin [60, 61] in 1981. Plotkin was the first to coin the terminology *structural operational semantics* (SOS), but what he meant by that was mostly what we call today *small-step structural operational semantics* (small-step SOS). Note, however, that Plotkin in fact used a combination of small-step and big-step SOS, without calling them as such, using the \rightarrow arrow for small-steps and its transitive closure \rightarrow^* for big-steps. We will discuss small-step SOS in depth in Section 3.3. Kahn and others found big-step SOS more natural and convenient than Plotkin’s SOS, essentially because it is more abstract

and denotational in nature (which may help in formal reasoning), and one needs fewer rules to define a language semantics.

One of the most notable uses of natural semantics is the formal semantics of Standard ML by Milner *et al.* [49]. Several types of big-step sequents were used in [49], such as $\rho \vdash p \Rightarrow v/f$ for “in environment ρ , sentence p either evaluates to value v or otherwise an error or failure f takes place”, and $\sigma, \rho \vdash p \Rightarrow v, \sigma'$ for “in state σ and environment ρ , sentence p evaluates to v and the resulting state is σ' ”, and $\rho, v \vdash m \Rightarrow v'/f$ for “in environment ρ , a match m either evaluates to v' or otherwise failure f ”, among many others. After more than twenty years of natural semantics, it is now common wisdom that big-step semantics is inappropriate as a rigorous formalism for defining languages with complex features such as exceptions or concurrency. To give a reasonably compact and readable definition of Standard ML in [49], Milner *et al.* had to make several informal notational conventions, such as a “state convention” to avoid having to mention the state in every rule, and an “exception convention” to avoid having to more than double the number of rules for the sole purpose of supporting exceptions. As rightfully noticed by Mosses [53], such conventions are not only adhoc and language specific, but may also lead to erroneous definitions. Section 3.5 illustrates in detail the limitations of big-step operational semantics, both with respect to its incapacity of defining certain rather simple language features and with respect to inconvenience in using it (for example, due to its lack of modularity). One of the common uses of natural semantics these days is to define static semantics of programming languages and calculi, such as type systems (see Section 3.2.4).

Hennessy [32] (1990) and Winskel [87] (1993) are perhaps the first textbooks proposing big-step SOS in teaching programming language semantics. They define big-step SOS for several simple languages, including ones similar to the IMP language presented in this chapter. Hennessy [32] defines languages incrementally, starting with a small core and then adding new features one by one, highlighting a major problem with big-step SOS: its lack of modularity. Indeed, the big-step SOS of a language is entirely redefined several times in [32] as new features are added to the language, because adding new features requires changes in the structure of judgments. For example, some big-step SOS judgments in [32] evolve from $a \Rightarrow i$, to $\sigma \vdash a \Rightarrow i$, to $D, \sigma \vdash i$ during the language design experiment, where a is an expression, i an integer, σ a state (or environment), and D a set of function definitions.

While the notations of Hennessy [32] and of Milner *et al.* [49] are somehow reminiscent of original Kahn’s notation, Winskel [87] uses a completely different notation. Specifically, he prefers to use big-step sequents of the form $\langle a, \sigma \rangle \rightarrow i$ instead of $\sigma \vdash a \Rightarrow i$. There seems to be, unfortunately, no uniform and/or broadly accepted notation for SOS sequents in the literature, be they big-step or small-step. As already explained earlier in this section, for the sake of uniformity at least throughout this book, we will make an effort to consider sequents of the form $C \Downarrow R$ in our big-step SOS definitions, where C and R are configurations. Similarly, we will make an effort to use the notation $C \rightarrow C'$ for small-step sequents (see Section 3.3). We will make it explicit when we deviate from our uniform notation, explaining how the temporary notation relates to the uniform one, as we did in Section 3.2.4.

Big-step SOS is the semantic approach which is probably the easiest to implement in any language or to represent in any computational logic. There are countless approaches to implementing or encoding big-step SOS in various languages or logics, which we cannot enumerate here. We only mention rewriting-based ones which are directly related to the approach followed in this book. Vardejo and Martí-Oliet [84] proposed big-step SOS implementations in Maude for several languages, including Hennessy’s languages [32] and Kahn’s Mini-ML [34]. Vardejo and Martí-Oliet were mainly interested in demonstrating the strengths of Maude 2 to give executable semantics to concrete languages, rather than in proposing general representations of big-step SOS into rewrite logic that work for any language. Besides Vardejo and Martí-Oliet, several other authors used rewrite logic and Maude to define and implement language semantics for languages or calculi

following a small-step approach. These are discussed in Section 3.3.4; we here only emphasize that most of those can likely be adapted into big-step SOS definitional or implementation styles, because big-step SOS can be regarded as a special case of small-step SOS, one in which the small step is “big”.

There is a common misconception that big-step SOS is “efficient” when executed. This misconception is fueled by case studies where efficient interpreters for various (deterministic or deterministic fragments of) languages, were more or less mechanically derived from the big-step SOS of those languages. Recall that a big-step SOS is a formal proof system, and that proof systems are typically meant to tell *what* is possible in the defined language, calculus or system, and not *how* to implement it. A big-step SOS *can* indeed be used as a basis to develop efficient interpreters, but one should be aware of the fact that when that happens it happens either because the big-step SOS has a particularly convenient form, where at most one proof rule can apply at any given moment⁴, or because one cuts corners in the implementation by deliberately⁵ ignoring the proof search process needed to detect which proof rule applies, and instead arbitrarily picking one matching rule and stopping the execution if its premises cannot be proved. Section 3.5 illustrates a big-step SOS whose rules are not syntax-driven (e.g., expressions have side effects and arithmetic operators are non-deterministic); as discussed there, its faithful Maude implementation is indeed very slow, requiring exponential complexity in some cases to derive the rule premises.

3.2.6 Exercises

Prove the following exercises, all referring to the IMP big-step SOS in Figure 3.7.

Exercise 54. *Change the rule (BIGSTEP-DIV) so that division short-circuits when a_1 evaluates to 0. (Hint: may need to replace it with two rules, like for the semantics of conjunction).*

Exercise 55. *Change the big-step semantics of the IMP conjunction so that it is not short-circuited.*

Exercise 56. *Add an “error” state and modify the big-step semantics in Figure 3.7 to allow derivations of sequents of the form $\langle s, \sigma \rangle \Downarrow \langle \text{error} \rangle$ or $\langle p \rangle \Downarrow \langle \text{error} \rangle$ when s evaluated in state σ or when p evaluated in the initial state performs a division by zero.*

Exercise 57. *Prove that the transition relation defined by the BIGSTEP(IMP) proof system in Figure 3.7 is **deterministic**, that is:*

- *If $\text{BIGSTEP(IMP)} \vdash \langle a, \sigma \rangle \Downarrow \langle i \rangle$ and $\text{BIGSTEP(IMP)} \vdash \langle a, \sigma \rangle \Downarrow \langle i' \rangle$ then $i = i'$;*
- *If $\text{BIGSTEP(IMP)} \vdash \langle b, \sigma \rangle \Downarrow \langle t \rangle$ and $\text{BIGSTEP(IMP)} \vdash \langle b, \sigma \rangle \Downarrow \langle t' \rangle$ then $t = t'$;*
- *If s terminates in σ then there is a unique σ' such that $\text{BIGSTEP(IMP)} \vdash \langle s, \sigma \rangle \Downarrow \langle \sigma' \rangle$;*
- *If p terminates then there is a unique σ such that $\text{BIGSTEP(IMP)} \vdash \langle p \rangle \Downarrow \langle \sigma \rangle$.*

Prove the same results above for the proof system detecting division-by-zero as in Exercise 56.

Exercise 58. *Show that there is no algorithm, based on the big-step proof system in Figure 3.7 or on anything else, which takes as input an IMP program and says whether it terminates **or not**.*

⁴We call such big-step SOS definitions *syntax-driven*.

⁵Either because one proved that this is sound, or because one’s intention is to trade soundness for performance.